Anharmonic oscillators and generalized squeezed states

2008 J. Phys. A: Math. Theor. 41 304015

(http://iopscience.iop.org/1751-8121/41/30/304015)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **41** (2008) 304015 (10pp) doi:10.1088/1751-8113/41/30/304015

Anharmonic oscillators and generalized squeezed states

D Afshar and M Jafarpour

Physics Department, Shahid Chamran University, Ahvaz, Iran

E-mail: da_afshar@yahoo.com and mojtaba_jafarpour@hotmail.com

Received 11 October 2007, in final form 14 February 2008
Published 15 July 2008
Online at stacks.iop.org/JPhysA/41/304015

Abstract
We present a new method to study the eigenvalues and eigenfunctions of anharmonic oscillators. It involves a new class of Bogoliubov transformations and leads to the introduction of k-photon coherent states. We consider the Hamiltonians for the simple harmonic and anharmonic oscillators as the two generators of a Lie algebra, whose other generators may be found exactly, or up to any desired order of the perturbation parameter involved. An element of this Lie group, turning out to be the multi-photon operator, transforms the anharmonic Hamiltonian to the harmonic one, thus facilitating the calculation of the eigenvalues and eigenfunctions of the former. The transformation of the ordinary annihilation and creation operators leads to generalized ones, corresponding to generalized oscillation modes, and also helps us out to introduce multi-photon coherent states. We specifically consider four-photon coherent states in detail and study time dependent position and momentum uncertainties in these states.

PACS numbers: 42.50.Dv, 03.65.Ge, 02.20.Sv

1. Introduction

Simple harmonic oscillator is an idealized model to describe many phenomena in physics and chemistry. Anharmonic oscillator is a deviation from this idealized model to a realistic one. Rayleigh–Schrödinger perturbation theory has been widely used, providing the energy of an anharmonic oscillator as a formal power series of the perturbation parameter involved [1–4]. However, the power series diverges even for small coupling constants; thus, appropriate techniques must be applied to alleviate this problem [5–12]. Several alternative approaches have also been developed; among them, a Lie algebraic method using canonical transformations [13], multiple-scale method [14–16], operator method [17], quasilinearization method [18, 19], a variational method based on the squeezed states [20, 21] and Naundorf method [22] are mentioned. We also refer the reader to the book by Fernandez and Castro for a detailed exposition of algebraic methods [23].

We also note that nonlinear optical processes occur, when fields interact with the kind of matter that is characterized by nonlinear properties and can be described by anharmonic oscillators [24, 25]. Such processes correspond to the production of two- or k-photon coherent states [26, 27]. One expects to generalize the squeezed states or two-photon coherent states to k-photon ones, by means of straightforward generalization of the two-photon unitary operator; however, this task encounters divergence difficulties [28], which may partially be overcome by using the Padé approximation techniques [29]. An approach to this generalization, based on the Brandt–Greenberg multi-photon operators, has also been used [30–33]. Furthermore, we mention a recent method, based on the generalization of the Bogoliubov transformations [34, 35].

In this work, we present a new simple method to find the eigenvalues of anharmonic oscillators. It also bridges between the notion of the latter and the generalized squeezing. We use Lie algebraic methods to relate the Hamiltonian of an anharmonic oscillator to that of a harmonic one, in a perturbation sense. This is achieved via a canonical transformation, which is unique to the structure of the specific Hamiltonian under study. We introduce our method in section 2, where we apply it to two prototype trivial examples: a harmonic oscillator with a linear perturbation term and one with a quadratic perturbation term that just brings about a frequency shift.

In section 3, a unitary transformation is found that relates the Hamiltonian of a quartic anharmonic oscillator to that of a harmonic one, and it is realized as a generalized squeezing operator. The eigenvalues of the anharmonic oscillator are evaluated up to the fourth order in the perturbation parameter, and the results are compared with those obtained from Rayleigh–Schrödinger perturbation theory. Section 4 will be devoted to the study of four-photon coherent states that emerge from our method. Time dependent position and momentum uncertainties are also obtained in these states and their oscillating nature is studied. In section 5, we study the quartic-quadratic anharmonic oscillator and introduce another type of four-photon coherent states. Finally, we deal with the discussion and conclusions in section 6.

2. The Lie algebra method

The anharmonic oscillator, described by the Hamiltonian

$$H_n = H_0 + \lambda x^n = \frac{p^2}{2} + \frac{x^2}{2} + \lambda x^n, \tag{1}$$

is a deviation from the harmonic one described by the Hamiltonian H_0. If N Hermitian generators L_i, including $L_1 = H_0$ and $L_2 = H_n$, satisfying

$$[L_i, L_j] = \sum_k c_{ijk} L_k, \tag{2}$$

form a closed commutator algebra, the operators L_i specify a Lie group, whose generators they are. The elements of this group are given by

$$U = \exp\left(-\mathrm{i}\sum_{i=1}^{N} \alpha_i L_i\right), \tag{3}$$

where α_i's are real parameters. Now an element of the group may be found that transforms H_0 to H_n. This coming about, the eigenvalues and the eigenfunctions of the anharmonic oscillator can be obtained in terms of those of a harmonic one in a simple manner.

Now, let us assume $n = 1$ in (1) to write

$$H_1 = \frac{p^2}{2} + \frac{x^2}{2} + \lambda x = a^\dagger a + \frac{1}{2} + \lambda\left(\frac{a + a^\dagger}{\sqrt{2}}\right), \tag{4}$$

which may be considered as a harmonic oscillator in an external electric field. a and $a^\dagger$ are the ordinary annihilation and creation operators. Considering the commutator $[H_0, H_1] = \lambda(a^\dagger - a)/\sqrt{2}$, we realize that the operators H_0, H_1, $\mathrm{i}(a^\dagger - a)$ and the identity operator I form a closed commutator algebra and are the generators of a Lie group. The Glauber operator $U_1 = \exp(\alpha a^\dagger - \alpha^* a)$, with α assumed real in this section, is an element of this group. The transformation of H_0 to H_1 is carried out in the following manner

$$H_1 = U_1^\dagger H_0 U_1 - \frac{\lambda^2}{2}, \tag{5}$$

where α has been replaced by $\lambda/\sqrt{2}$ in U_1. Using (5), we find

$$H_1\left[U_1^\dagger|n\rangle_a\right] = \left[\left(n + \frac{1}{2}\right) - \frac{\lambda^2}{2}\right]\left[U_1^\dagger|n\rangle_a\right],$$

where $\{|n\rangle_a\}$ are a-mode number states; meaning that the eigenvalues of H_1 are given by

$$E_n = \left(n + \frac{1}{2}\right) - \frac{\lambda^2}{2}, \tag{6}$$

and the eigenfunctions, called the displaced number states [36] are expressed by

$$U_1^\dagger|n\rangle_a = \mathrm{e}^{\frac{\lambda}{\sqrt{2}}(a - a^\dagger)}|n\rangle_a. \tag{7}$$

Now, we consider our second prototype trivial example. The Hamiltonian of a shifted-frequency harmonic oscillator, the case $n = 2$ in (1), is given by

$$H_2 = \frac{p^2}{2} + \frac{x^2}{2} + \lambda x^2 = a^\dagger a + \frac{1}{2} + \lambda\left(\frac{a + a^\dagger}{\sqrt{2}}\right)^2. \tag{8}$$

In this case, we have $[H_0, H_2] = \lambda(a^{\dagger 2} - a^2)$; therefore, the Hermitian operators $H_0, H_2, \mathrm{i}(a^{\dagger 2} - a^2)$ and I, form a closed commutator algebra and are the generators of a Lie group. The squeezing operator $U_2 = \exp(\beta a^{\dagger 2} - \beta^* a^2)$, with β assumed real, is an element of this group. The transformation of the fundamental mode operators a and $a^\dagger$ into the new ones, b and $b^\dagger$ under U_2, is carried out by the following Bogoliubov transformations:

$$b = U_2^\dagger a U_2 = \mu a + \nu a^\dagger, \qquad b^\dagger = U_2^\dagger a^\dagger U_2 = \mu a^\dagger + \nu a,$$

where $\mu = \cosh(2\beta)$ and $\nu = \sinh(2\beta)$. As $\mu^2 - \nu^2 = 1$, we have also $[b, b^\dagger] = 1$. The transformed Hamiltonian under U_2 is given by

$$H_2 = \sqrt{1 + 2\lambda}\, U_2^\dagger H_0 U_2 = \sqrt{1 + 2\lambda}\left(b^\dagger b + \tfrac{1}{2}\right), \tag{9}$$

where $\lambda = 2\mu\nu/(\mu - \nu)^2$ has been assumed. Using (9), we find

$$H_2\left[U_2^\dagger|n\rangle_a\right] = \sqrt{1 + 2\lambda}\left(n + \tfrac{1}{2}\right)\left[U_2^\dagger|n\rangle_a\right],$$

or

$$H_2|n\rangle_b = \sqrt{1 + 2\lambda}\left(n + \tfrac{1}{2}\right)|n\rangle_b,$$

implying that eigenfunctions of the shifted-frequency harmonic oscillator, the b-mode number states, or squeezed number states [36], are given by $|n\rangle_b = U_2^\dagger|n\rangle_a$, and the corresponding energies by $\sqrt{1 + 2\lambda}\left(n + \frac{1}{2}\right)$.

3. Quartic anharmonic oscillator

Now we consider the quartic anharmonic oscillator. Its Hamiltonian in terms of the fundamental mode operators a and $a^\dagger$, may be given by

$$H_4 = a^\dagger a + \frac{1}{2} + \lambda\left(\frac{a + a^\dagger}{\sqrt{2}}\right)^4. \tag{10}$$

We can show that

$$[H_0, H_4] = -\mathrm{i}L_3 - 3\mathrm{i}L_4 - 2\mathrm{i}L_5, \tag{11}$$

where

$$L_3 = \mathrm{i}\lambda(a^{\dagger 4} - a^4), \qquad L_4 = \mathrm{i}\lambda(a^{\dagger 2} - a^2), \qquad L_5 = \mathrm{i}\lambda(a^{\dagger 3}a - a^\dagger a^3).$$

Inspecting the mutual commutators between the operators H_0, H_4, L_3, L_4 and L_5, we are finally led to prove that $H_0, H_4, \mathrm{i}\lambda(a^{\dagger 4} - a^4), \mathrm{i}\lambda(a^{\dagger 2} - a^2), \mathrm{i}\lambda(a^{\dagger 3}a - a^\dagger a^3), \lambda(a^{\dagger 4} + a^4)$, $\lambda(a^{\dagger 2} + a^2), \lambda(a^{\dagger 3}a + a^\dagger a^3)$ and I form a closed commutator algebra up to the first order if the perturbation parameter λ is small; therefore, they are the generators of a Lie group, up to that order. We now focus on the following element of this group, known as the four-photon operator [28]

$$U_4 = \exp[A\lambda(a^{\dagger 4} - a^4) + B\lambda(a^{\dagger 2} - a^2) + C\lambda(a^{\dagger 3}a - a^\dagger a^3)], \tag{12}$$

where A, B and C are real parameters. This operator transforms the fundamental mode operators, a and $a^\dagger$ to the new ones c and $c^\dagger$. Using Baker–Campbell–Hausdorff relation up to the first order in the parameter λ, we have

$$\begin{aligned} c &= U_4^\dagger a U_4 \\ &= a + \lambda(4Aa^{\dagger 3} + 2Ba^\dagger + 3Ca^{\dagger 2}a - Ca^3), \end{aligned} \tag{13}$$

$$\begin{aligned} c^\dagger &= U_4^\dagger a^\dagger U_4 \\ &= a^\dagger + \lambda(4Aa^3 + 2Ba + 3Ca^\dagger a^2 - Ca^{\dagger 3}). \end{aligned} \tag{14}$$

It is worth noting that c and $c^\dagger$ also obey the canonical commutation relation $[c, c^\dagger] = 1$, up to the first order in the parameter λ; therefore, we have introduced a new class of Bogoliubov transformations by (13) and (14). Assuming $A = 1/16$, $B = 3/4$ and $C = 1/2$, the four-photon operator transforms H_0 to H_4, up to the first order in the parameter λ, as follows

$$\begin{aligned} U_4^\dagger H_0 U_4 &= c^\dagger c + \frac{1}{2} = H_4 - \frac{3\lambda}{2}\left(N_a^2 + N_a\right) - \frac{3\lambda}{4}, \\ H_0 &= U_4 H_4 U_4^\dagger - \frac{3\lambda}{2}N_a(N_a + 1) - \frac{3\lambda}{4}, \end{aligned} \tag{15}$$

where $N_a = a^\dagger a$ is the normal number operator. Using (15), we find

$$H_4\left[U_4^\dagger|n\rangle_a\right] = \left[n + \frac{1}{2} + \frac{3\lambda}{4} + \frac{3\lambda}{2}n(n+1)\right]\left[U_4^\dagger|n\rangle_a\right].$$

Thus, the eigenstates of the quartic anharmonic oscillator, our c-mode number states, are given by

$$\begin{aligned} |n\rangle_c &\equiv U_4^\dagger|n\rangle_a, \\ |n\rangle_c &= \mathrm{e}^{-\lambda[\frac{1}{16}(a^{\dagger 4} - a^4) + \frac{3}{4}(a^{\dagger 2} - a^2) + \frac{1}{2}(a^{\dagger 3}a - a^\dagger a^3)]}|n\rangle_a. \end{aligned} \tag{16}$$

This result is formally correct up to the order λ and corresponds to the first-order wavefunction calculations in Rayleigh–Schrödinger and multiple-scale perturbation theories [15, 37]. The first-order perturbed energies are expressed by

$$E_n = n + \frac{1}{2} + \frac{3\lambda}{4} + \frac{3\lambda}{2} n(n+1),$$

which are also in complete agreement with those obtained, using the first-order Rayleigh–Schrödinger perturbation theory [37].

Furthermore, although (16) is formally derived up to the first order in the parameter λ, we can derive the energy eigenvalues, using these states, up to higher orders of accuracies. To show this, we calculate the expectation values of H_4 in the c-mode number state basis, up to the fourth order and compare the energies obtained, with the standard results. We find

$$E_0 = 0.5 + 0.75\lambda - 2.625\lambda^2 + 20.8125\lambda^3 - 104.098\lambda^4 + \cdots, \tag{17}$$

$$E_1 = 1.5 + 3.75\lambda - 20.625\lambda^2 + 244.688\lambda^3 - 1628.96\lambda^4 + \cdots, \tag{18}$$

$$E_2 = 2.5 + 9.75\lambda - 76.875\lambda^2 + 1254.94\lambda^3 - 10791\lambda^4 + \cdots, \tag{19}$$

$$E_3 = 3.5 + 18.75\lambda - 196.875\lambda^2 + 4176.56\lambda^3 - 44972.4\lambda^4 + \cdots. \tag{20}$$

Bender and Wu also give the following perturbation series, for the ground-state energy $E_{0,B}$, of the quartic anharmonic oscillator which we write down up to the fourth order as follows: [1]

$$E_{0,B} = 0.5 + 0.75\lambda - 2.625\lambda^2 + 20.8125\lambda^3 - 241.289\lambda^4 + \cdots. \tag{21}$$

We note that the result (17) agrees with (21), up to the third order in the parameter λ. We have also found out that our results (17) through (20) are compatible with those obtained from multiple-scale method, up to the third order in the perturbation parameter [16]. Moreover, table 1 in [17] displays the energy levels derived from the operator method and also from the Rayleigh–Schrödinger perturbation theory which are in good agreement with ours as well.

The agreements we have already observed should not surprise the reader; this situation is similar to the one we encounter in Rayleigh–Schrödinger perturbation theory. That is, calculation of the $(n + 1)$ th-order energy shift requires only the knowledge of the nth-order wavefunction. However, if all the perturbed wavefunctions to the order n are known, all the energy eigenvalues up to the order $(2n + 1)$ can be obtained. Moreover, both series are divergent, even for small values of λ; specifically, we observe that $\langle 0|U_4|0\rangle$ is divergent [28], and thus responsible for the divergence of (17). Therefore, one has to truncate the series to get any physical result. Moreover, the accuracy that can be achieved depends on the order of truncation and the magnitude of the coupling constant λ [12]. For example, using (17), we calculate the ground-state energy up to the fourth order; for $\lambda = 0.1$ we find $E_0 = 0.5591527$. Comparing this result with the exact value $E_0 = 0.559146327183519576$ obtained by Vinette and Čížek [38], we find that the error is about 0.001%.

4. Four-photon coherent states

Ordinary coherent states, or the so-called one-photon coherent states, are eigenstates of the ordinary annihilation operator a

$$a|\alpha\rangle = \alpha|\alpha\rangle, \qquad |\alpha\rangle = U_1(\alpha)|0\rangle_a = \mathrm{e}^{\alpha a^\dagger - \alpha^* a}|0\rangle_a, \tag{22}$$

entailing a Poisson distribution in terms of the ordinary number states.

Defining b-mode vacuum state $|0\rangle_b$, by the relation $b|0\rangle_b = 0$, we may construct the so-called b-mode number states by the repeated application of b-mode creation operator $b^\dagger$ on the b-mode vacuum state as follows

$$|n\rangle_b = \frac{b^{\dagger n}}{\sqrt{n!}}|0\rangle_b, \qquad |n\rangle_b \equiv U_2^\dagger |n\rangle_a. \tag{23}$$

Eigenstates of the operator b, $|\beta\rangle$ are called the b-mode coherent states, two-photon coherent states, or the familiar squeezed states

$$b|\beta\rangle = \beta|\beta\rangle, \qquad |\beta\rangle = U_2^\dagger U_1(\beta)|0\rangle_a = \mathrm{e}^{\beta b^\dagger - \beta^* b}|0\rangle_b. \tag{24}$$

In a similar way, we may introduce the c-mode number states, by the relation

$$|n\rangle_c = \frac{c^{\dagger n}}{\sqrt{n!}}|0\rangle_c, \qquad |n\rangle_c \equiv U_4^\dagger |n\rangle_a, \tag{25}$$

where $|0\rangle_c$ is the c-mode vacuum state and is defined by $c|0\rangle_c = 0$. The four-photon or c-mode coherent states, or the generalized squeezed states $|\gamma\rangle$, are defined as the eigenstates of the c-mode annihilation operator as follows

$$c|\gamma\rangle = \gamma|\gamma\rangle, \tag{26}$$

where γ is a complex number. Using (13), we have

$$U_4^\dagger a U_4|\gamma\rangle = \gamma|\gamma\rangle, \qquad a(U_4|\gamma\rangle) = \gamma(U_4|\gamma\rangle).$$

Thus, $U_4|\gamma\rangle$ is an a-mode coherent state, and we have

$$U_4|\gamma\rangle = U_1(\gamma)|0\rangle_a. \tag{27}$$

Therefore, the four-photon coherent states are given by

$$|\gamma\rangle = U_4^\dagger(\lambda)U_1(\gamma)|0\rangle_a = \mathrm{e}^{\gamma c^\dagger - \gamma^* c}|0\rangle_c. \tag{28}$$

Using this result, we finally write down the four-photon coherent state $|\gamma\rangle$, up to the first order in the perturbation parameter λ, in terms of the ordinary number states, as follows

$$\begin{aligned}|\gamma\rangle = \mathrm{e}^{-|\gamma|^2/2}\sum_{n=0}\Bigg[\frac{\gamma^n}{\sqrt{n!}} + \lambda\Bigg(&-\frac{\gamma^{n-4}\sqrt{n!}}{16(n-4)!}\theta(n-4) + \frac{\gamma^{n+4}}{16\sqrt{n!}} - \frac{3\gamma^{n-2}\sqrt{n!}}{4(n-2)!}\theta(n-2)\\ &+ \frac{3\gamma^{n+2}}{4\sqrt{n!}} - \frac{\gamma^{n-2}\sqrt{n!}}{2(n-3)!}\theta(n-3) + \frac{\gamma^{n+2}n}{2\sqrt{n!}}\theta(n-1)\Bigg)\Bigg]|n\rangle_a,\end{aligned} \tag{29}$$

where $\theta(x)$ is equal to 1 for $x \geqslant 0$ and equal to 0 for $x < 0$.

Now we embark upon the derivation of uncertainties. The time dependent position operator for the quartic anharmonic oscillator, up to the first order in the parameter λ is given by [39]

$$x(t) = \tfrac{1}{\sqrt{2}}\{(\cos t - \mathrm{i}\sin t)a(0) - \lambda[A_1 a(0) + A_2 a^3(0) + A_1^* a^{\dagger 2}(0)a(0)]\} + \text{h.c.}, \tag{30}$$

where h.c. stands for the Hermitian conjugate of the other terms, and the parameters A_1 and A_2 are defined as follows

$$A_1 = \tfrac{3}{4}[t\sin t + \mathrm{i}(t\cos t - \sin t)], \qquad A_2 = \tfrac{1}{16}[(\cos t - \cos 3t) + \mathrm{i}(\sin 3t - 3\sin t)].$$

We also take the time derivative of (30) to find the momentum operator

$$p(t) = -\tfrac{1}{\sqrt{2}}\{(\sin t + \mathrm{i}\cos t)a(0) + \lambda[\dot{A}_1 a(0) + \dot{A}_2 a^3(0) + \dot{A}_1^* a^{\dagger 2}(0)a(0)]\} + \text{h.c.}, \tag{31}$$

where $\dot{A_1}$ and $\dot{A_2}$ are time derivatives of A_1 and A_2.

It is straightforward to obtain the position and momentum uncertainties for the four-photon coherent state $|\gamma\rangle$, by means of (30) and (31) as follows:

$$\begin{aligned}\Delta^2 x(t) = \frac{1}{2} - \frac{\lambda}{2}\Bigg\{\left(\frac{3}{2} + 3|\gamma|^2\right)\sin^2 t + 2\mathrm{Re}\Bigg[&\left(\frac{3}{4}\gamma^2 + 3|\gamma|^2 + \frac{3}{2}\right)\mathrm{e}^{2\mathrm{i}t}\\ &+ A_1\gamma^2\mathrm{e}^{-\mathrm{i}t} + 3A_2\gamma^2\mathrm{e}^{\mathrm{i}t}\Bigg]\Bigg\},\end{aligned} \tag{32}$$

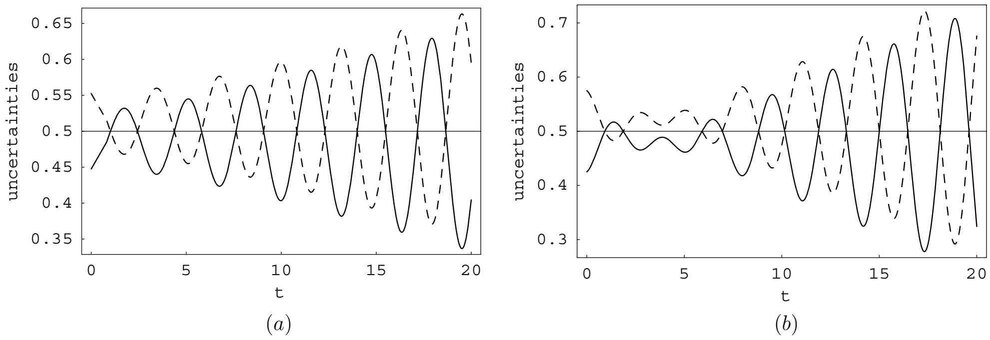

Figure 1. Plot of the $\hat{x}$-uncertainty (solid line) and $\hat{p}$-uncertainty (dashed line) for four-photon coherent state versus time: (a) $\lambda = 0.01$, $\gamma = 1$; (b) $\lambda = 0.01$ and $\gamma = 1 + \mathrm{i}$.

$$\Delta^2 p(t) = \frac{1}{2} + \frac{\lambda}{2}\left\{\left(\frac{3}{2} + 3|\gamma|^2\right)\sin^2 t + 2\mathrm{Re}\left[\left(\frac{3}{4}\gamma^2 + 3|\gamma|^2 + \frac{3}{2}\right)\mathrm{e}^{2\mathrm{i}t} + \mathrm{i}\dot{A}_1\gamma^2\mathrm{e}^{-\mathrm{i}t} - 3\mathrm{i}\dot{A}_2\gamma^2\mathrm{e}^{\mathrm{i}t}\right]\right\}, \tag{33}$$

where all the expressions of order two and higher in terms of λ are neglected. In the absence of the quartic term ($\lambda = 0$), the uncertainties are equal to $\frac{1}{2}$; just as it is the case for a coherent state or a vacuum number state.

The position and the momentum uncertainties have a simple oscillating behavior in time around $\frac{1}{2}$, for the c-mode vacuum state ($\gamma = 0$). The values change in the range $\frac{1}{2} - \frac{3\lambda}{2}$ and $\frac{1}{2} + \frac{3\lambda}{4}$ for position and in the range $\frac{1}{2} - \frac{3\lambda}{4}$ and $\frac{1}{2} + \frac{3\lambda}{2}$ for momentum. The uncertainty product is $\frac{1}{4}$, up to the first order; implying that the c-mode vacuum state is a minimum uncertainty state, up to the first order in λ.

In the general case, the product of uncertainties (32) and (33) reduces to $\frac{1}{4}$, up to the first order in λ, at $t \to 0$ limit. Thus, we conclude that the four-photon coherent state (29) is also a minimum uncertainty state at $t = 0$. As time passes, one of the uncertainties dips well more and more below the value $\frac{1}{2}$ in successive periods, thus revealing the squeezing properties of such states. We have illustrated the uncertainties of $\hat{x}$ and $\hat{p}$ versus time in figure 1. It is observed that these quantities oscillate in opposite directions and their overall range of variation increases with time.

5. Quadratic-quartic anharmonic oscillator

We now consider the quadratic-quartic anharmonic oscillator described by the Hamiltonian

$$H_q = \frac{p^2}{2} + \omega^2 x^2 + \lambda x^4. \tag{34}$$

Using (9) and (15), we can transform H_0 to H_q, up to the first order, by implementing the unitary transformation $Q = U_4(\Lambda)U_2(\beta)$, as follows

$$\begin{aligned} Q^\dagger H_0 Q &= \frac{1}{\sqrt{2}\omega}H_q - \frac{3\lambda}{4\sqrt{2}\omega^3}U_2^\dagger N_a(N_a+1)U_2 - \frac{3\lambda}{8\sqrt{2}\omega^3}, \\ H_0 &= \frac{1}{\sqrt{2}\omega}QH_qQ^\dagger - \frac{3\lambda}{4\sqrt{2}\omega^3}N_a(N_a+1) - \frac{3\lambda}{8\sqrt{2}\omega^3}, \end{aligned} \tag{35}$$

where $\beta = \ln(2\omega^2)/8$ and $\Lambda = \lambda/(2\sqrt{2}\omega^3)$ has been assumed. In view of (35), we have

$$H_q[Q^\dagger|n\rangle_a] = \sqrt{2}\omega\left\{n + \frac{1}{2} + \frac{3\lambda}{4\sqrt{2}\omega^3}\left[n(n+1) + \frac{1}{2}\right]\right\}[Q^\dagger|n\rangle_a],$$

implying that the eigenstates of H_q are expressed by $Q^\dagger|n\rangle_a = U_2^\dagger(\beta)U_4^\dagger(\Lambda)|n\rangle_a$ and its energy levels by

$$E_n = \sqrt{2}\omega\left\{n + \frac{1}{2} + \frac{3\lambda}{4\sqrt{2}\omega^3}\left[n(n+1) + \frac{1}{2}\right]\right\}. \tag{36}$$

To check the accuracy of the last result, we consider the first two energy levels, E_0 and E_1 as examples; for $\lambda = 0.01$ and $\omega^2 = 2$, (36) yields $E_0 = 1.001\,875$ and $E_1 = 3.009\,375$. Using the relatively accurate field theoretical method that we have developed in [20], we have obtained the values 1.001 867 and 3.009 311 for the same quantities respectively, showing that the discrepancies are less than 0.001% .

Now let us consider the d-mode annihilation and creation operators defined by the following transformations:

$$d = Q^\dagger a Q, \qquad d^\dagger = Q^\dagger a^\dagger Q.$$

These lead to

$$d|\eta\rangle = \eta|\eta\rangle, \qquad |\eta\rangle = \mathrm{e}^{\eta d^\dagger - \eta^* d}|0\rangle_d, \tag{37}$$

where $|0\rangle_d$ is the d-mode vacuum state. We also have

$$Q^\dagger a Q|\eta\rangle = \eta|\eta\rangle, \qquad a(Q|\eta\rangle) = \eta(Q|\eta\rangle).$$

Thus the states $Q|\eta\rangle$ are a-mode coherent states and we can write

$$Q|\eta\rangle = U_1(\eta)|0\rangle_a. \tag{38}$$

Therefore we have

$$|\eta\rangle = Q^\dagger U_1|0\rangle_a = U_2^\dagger U_4^\dagger U_1|0\rangle_a. \tag{39}$$

States $|\eta\rangle$, as expressed by (37) and (39) are another type of multi-photon coherent states.

6. Discussion

We have introduced a new and concise method to study the anharmonic oscillators, which also leads to the introduction of generalized squeezed states. The method is based on the construction of a Lie group, whose generators include H_0 and H_n. The generator of the group, H_0 may be transformed to the generator H_n by a unitary transformation, endowed by an element of the group. If a closed commutator algebra is formed exactly, the method can be used to obtain exact eigenvalues and eigenfunctions of H_n. However, the commutator algebra is generally obeyed up to some order of the parameter λ, thus the eigenvalues and eigenfunctions of H_n are found approximately, up to that order. Of course, if one extends the method to higher orders, the number of the generators increases and one has to perform more and more algebraic calculations, as happens in the case of any higher order perturbation theory. In the cases of H_1 and H_2, the familiar Glauber and squeezed operators, being canonical transformations, emerge respectively; while in the case of H_4, the four-photon operators are generated. The transformation of the annihilation operator a, under the squeezed operator U_2, leads to the Bogoliubov transformations and the associated squeezed states. Keeping terms up to the first order, the transformation of the ordinary annihilation operator under four-photon operator, leads to a new class of Bogoliubov transformations and the associated four-photon coherent states. While $N_a = a^\dagger a$ represents the number of ordinary photons, the transformed operators $N_b = b^\dagger b$ and $N_c = c^\dagger c$ represent the shifted frequency number operator and the quasi number operator, respectively. The effect of $N_c = c^\dagger c$ on the quasi number states $|n\rangle_c$ is similar to that of the ordinary number operator N_a on the normal number states $|n\rangle_a$. The four-photon coherent state is a Poisson distribution of quasi number states, as the coherent states and squeezed states are the Poisson distribution of ordinary and squeezed number states, respectively. The uncertainties for the four-photon coherent state, $|\gamma\rangle$, oscillate around $\frac{1}{2}$ in time similarly to the squeezed states, but the overall range of their variation grows with time. We also observed that $U_2^\dagger U_4^\dagger |n\rangle_a$ is an eigenstate of the Hamiltonian for the quadratic-quartic anharmonic oscillator and $U_2^\dagger U_4^\dagger U_1 |0\rangle_a$ represents another type of four-photon coherent state.

Finally, one can generalize the procedure to six photon coherent states via sextic anharmonic oscillator, eight photon coherent states via octic anharmonic oscillator and so on, in a similar manner.

Acknowledgments

The authors are greatful to the referees for their careful reading of the paper and their helpful comments.

References

[1] Bender C M and Wu T T 1969 *Phys. Rev.* **184** 1231–60
[2] Bender C M and Wu T T 1971 *Phys. Rev. Lett.* **27** 461–5
[3] Bender C M and Wu T T 1973 *Phys. Rev.* D **7** 1620–36
[4] Banks T, Bender C M and Wu T T 1973 *Phys. Rev.* D **8** 3346–66
[5] Seznec R and Zinn-Justin J 1979 *J. Math. Phys.* **20** 1398–408
[6] Fernández F M 1995 *J. Math. Phys.* **36** 3922–30
[7] Janke W and Kleinert H 1995 *Phys. Rev. Lett.* **75** 2787–91
[8] Weniger E J 1996 *Phys. Rev. Lett.* **77** 2859–62
[9] Ivanov I A 1996 *Phys. Rev.* A **54** 81–6
[10] Ivanov I A 1998 *J. Phys. A:Math. Gen.* **31** 6995–7003
[11] Skála L, Čížek J, Weniger E J and Zamastil J 1999 *Phys. Rev.* A **59** 102–6
[12] Weniger E J 1997 *Phys. Rev.* A **56** 5165–68
[13] Fernández F M 1992 *Phys. Rev.* A **45** 1333–8
[14] Bender C M and Bettencourt L M A 1996 *Phys. Rev. Lett.* **77** 4114–7
[15] Bender C M and Bettencourt L M A 1996 *Phys. Rev.* D **54** 7710–23
[16] Auberson G and CapdequiPeyranere M 2002 *Phys. Rev.* A **65** 032120
[17] Feranchuk I D, Komarov L I, Nichipor I V and Ulyanenkov A P 1995 *Ann. Phys., NY* **238** 370–440
[18] Mandelzweig V B 1999 *J. Math. Phys.* **40** 6266–91
[19] Liverts E Z, Mandelzweig V B and Tabakin F 2006 *J. Math. Phys.* **47** 062109
[20] Jafarpour M and Afshar D 2002 *J. Phys. A:Math. Gen.* **35** 87–92
[21] Jafarpour M and Afshar D 2005 *Nuovo Cimento* B **120** 335–44
[22] Gómez F J and Sesma J 2005 *J. Phys. A:Math. Gen.* **38** 3193–202
[23] Fernandez F M and Castro E A 1996 *Algebraic Methods in Quantum Chemistry and Physics* (Boca Raton, FL: CRC Press)
[24] Bondurant R S, Kumar P, Shapiro J H and Maeda M 1984 *Phys. Rev.* A **30** 343–53
[25] Levenson M D, Shelby R M, Aspect A, Reid M and Walls D F 1985 *Phys. Rev.* A **32** 1550–62
[26] Slusher R E, Hollberg L W, Yurke B, Mertz J C and Valley J F 1985 *Phys. Rev.* A **31** 3512–5
[27] Slusher R E, Hollberg L W, Yurke B, Mertz J C and Valley J F 1985 *Phys. Rev. Lett.* **55** 2409–12
[28] Fisher R A, Nieto M M and Sandberg V D 1984 *Phys. Rev.* D **29** 1107–10
[29] Braunstein S L and McLachlan R I 1987 *Phys. Rev.* A **35** 1659–67

[30] Brandt R A and Greenberg O W 1969 *J. Math. Phys.* **10** 1168–76
[31] D'Ariano G, Rasetti M and Vadacchino M 1985 *Phys. Rev.* D **32** 1034–37
[32] D'Ariano G M and Sterpi N 1989 *Phys. Rev.* A **39** 1860–68
[33] Bužek V and Jex I 1990 *Phys. Rev.* A **41** 4079–82
[34] DeSiena S, DiLisi A and Illuminati F 2001 *Phys. Rev.* A **64** 063803
[35] Wu Y and Côté R 2002 *Phys. Rev.* A **66** 025801
[36] Nieto M M 1997 *Phys. Lett.* A **229** 135–43
[37] McRae S M and Vrscay E R 1997 *J. Math. Phys.* **38** 2899–921
[38] Vinette F and Čížek J 1991 *J. Math. Phys.* **32** 3392–404
[39] Pathak A and Mandal S 2001 *Phys. Lett.* A **286** 261–76